Geometry

Full Syllabus

Illustrative Mathematics® 9-12 AGA (v.360)

8 Units · Unit Goals · Section Goals · Lesson-by-Lesson Outline

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Course Overview

Geometry of the Illustrative Mathematics 9-12 AGA curriculum is organized into 8 units. Students begin with constructions and rigid transformations, then prove triangles and quadrilaterals congruent. They study similarity, including similar right triangles and the Pythagorean Theorem, then build right-triangle trigonometry. They reason about solids—cross-sections, scaling, and volume—before turning to coordinate geometry, where they prove theorems algebraically and work with equations of circles and parabolas. They study the geometry of circles (chords, arcs, inscribed angles, tangents, sectors, and radians), and they close the year with conditional probability.

The unit sequence is:

- Unit 1: Constructions and Rigid Transformations
- Unit 2: Congruence
- Unit 3: Similarity
- Unit 4: Right Triangle Trigonometry
- Unit 5: Solid Geometry
- Unit 6: Coordinate Geometry
- Unit 7: Circles
- Unit 8: Conditional Probability

Each unit is divided into lettered sections (A, B, C, D, E as applicable), and each section is a sequence of daily lessons. Optional lessons are noted.

Unit 1: Constructions and Rigid Transformations

Unit Goal

Students perform compass-and-straightedge constructions, define rigid motions (reflections, translations, rotations) without reference to a coordinate grid, describe symmetries, and use transformations to prove theorems about lines, angles, and parallel lines.

Unit Narrative

This unit begins with constructions, continues to rigid transformations, and concludes with an introduction to proof writing. In grade 8, students determined the angle-preserving and length-preserving properties of rigid transformations experimentally, mostly with the help of a coordinate grid. Students have also previously studied the angle properties that they will prove in this unit.

Constructions play a significant role in the logical foundation of geometry. A focus of this unit is for students to explore properties of shapes in the plane without the aid of given measurements. At this point, students have worked so much with numbers, equations, variables, coordinate grids, and other quantifiable structures that it may come as a surprise just how far they can push concepts in geometry without measuring distances or angles.

At the beginning of the unit, students have the opportunity to move from informal explorations of lines and arcs to generating conjectures and writing justifications based on their constructions. The definition of a circle is an important foundation for concepts in this unit and throughout the course.

Next, students recall transformations from previous grades and learn to create rigid motions using construction tools instead of a grid. This leads to rigorous definitions of rotations, reflections, and translations without reference to a coordinate grid.

Finally, students begin to use the definitions they have learned to prove theorems. They begin to express their reasoning more formally, moving from vague statements ("It looks like . . .") toward the more formal point-by-point transformation proofs used by mathematicians.

Starting in the second section, a blank reference chart is provided for students, and a completed reference chart is provided for teachers. The reference chart is a resource for students to refer to as they make formal arguments. Students will continue adding to it throughout the course. Refer to the Course Guide for more information.

Section A: Constructions

Section Goals

- Create a construction from instructions.
- Describe construction steps precisely.
- Use circles in a construction to reason about lengths in figures.

Section Narrative

In this section, students use rigid compasses and straightedges to develop a catalog of construction techniques that they can use to construct a variety of figures. Students begin by following instructions and making observations about figures. Then they write their own instructions and identify ways to attend to precision in their communication. Multiple lessons include perpendicular lines. This key building block allows students to construct parallel lines and squares in this section. The perpendicular bisector is used for the definition of the term "reflection" later in this unit and in the proof of the Side-

Side-Side Triangle Congruence Theorem in a subsequent unit. Each construction technique supports students in refining their definitions of geometric objects and their ability to explain their thinking. In the culminating lesson, students build on their experiences with perpendicular bisectors to answer questions about allocating resources in a real-world situation.

Lessons

1. **Build It** — Let's use tools to create shapes precisely.
2. **Constructing Patterns** — Let's use compass and straightedge constructions to make patterns.
3. **Construction Techniques 1: Perpendicular Bisectors** — Let's explore equal distances.
4. **Construction Techniques 2: Equilateral Triangles** — Let's identify what shapes are possible within the construction of a regular hexagon.
5. **Construction Techniques 3: Perpendicular Lines and Angle Bisectors** — Let's use tools to solve some construction challenges.
6. **Construction Techniques 4: Parallel and Perpendicular Lines** — Let's use tools to draw parallel and perpendicular lines precisely.
7. **Construction Techniques 5: Squares** — Let's use straightedge and compass moves to construct squares.
8. **Using Technology for Constructions (Optional)** — Let's use technology to construct a diagram.
9. **Speedy Delivery** — Let's use perpendicular bisectors.

Section B: Defining Rigid Transformations

Section Goals

- Comprehend that rigid transformations produce congruent figures by preserving distance and angles.
- Draw the result of a transformation of a given figure.

Section Narrative

In this section, students study transformations both on and off of grids. Providing the structure of a grid allows students to make connections with their learning in previous courses. Then students learn precise definitions for the rigid motions, translations, reflections, and rotations that apply off the grid. Students recall that the figures are called congruent if there is a rigid transformation that takes one figure to the other. They examine the idea that rigid motions preserve distances and angles in a variety of contexts. In subsequent sections and units, students use the precise definitions generated here to prove theorems. This section intentionally allows extra time for students to learn new routines and establish norms for the year.

Lessons

1. **Rigid Transformations** — Let's draw some transformations.
2. **Defining Reflections** — Let's reflect some figures.
3. **Defining Translations** — Let's translate some figures.
4. **Incorporating Rotations** — Let's draw some transformations.
5. **Defining Rotations** — Let's rotate shapes precisely.

Section C: Working with Rigid Transformations

Section Goals

- Describe the symmetries of a figure using transformations that take the figure onto itself.
- Explain a sequence of transformations to take a given figure onto another.

Section Narrative

In this section, students examine the rigid transformations that take some shapes to themselves, otherwise known as symmetries. Then students continue describing sequences of rigid transformations that take one figure to another, congruent, figure. To prepare students for future congruence proofs, students start to come up with a systematic, point-by-point sequence of transformations that will work to take any pair of congruent polygons onto one another. There is an optional lesson if students need additional practice with this strategy. The point-by-point perspective also illustrates the transition from thinking about transformations as "moves" on the grid to thinking about transformations as functions that take points as inputs and produce points as outputs. The concept of transformations as functions is developed further in a later unit that explores coordinate geometry. This section intentionally allows extra time for students to learn new routines and establish norms for the year.

Lessons

1. **Symmetry** — Let's describe some symmetries of shapes.
2. **More Symmetry** — Let's describe more symmetries of shapes.
3. **Working with Rigid Transformations** — Let's compare transformed figures.
4. **Practicing Point-by-Point Transformations (Optional)** — Let's figure out some transformations.

Section D: Evidence and Proof

Section Goals

- Label diagrams in order to write precise conjectures.
- Prove theorems about lines and angles.

Section Narrative

In this section students create conjectures about angle relationships and prove them using what they know about rigid transformations. The primary work is on proof-writing skills. In order to focus on justification, many proofs are for ideas students were first introduced to in previous grades, such as supplementary, complementary, vertical, and adjacent angles. Students also prove the Triangle Angle Sum Theorem using the rigorous definitions from the previous section and a few assertions. Students are learning ways to express their reasoning more formally. Expect students to put together explanations with various levels of formality. Mastery of proof writing is not expected by the end of this section. A more reasonable goal is for students to consistently label and mark figures to indicate congruence which helps them communicate more precisely. The proofs in these materials are all written in narrative form. The narrative format matches the discussion students might have to use to convince their partner, and it also matches the way mathematicians write proofs.

Lessons

1. **Evidence, Angles, and Proof** — Let's make convincing explanations.
2. **Transformations, Transversals, and Proof** — Let's prove statements about parallel lines.
3. **One Hundred Eighty** — Let's prove the Triangle Angle Sum Theorem.

Section E: Let’s Put It to Work

Section Narrative

In this final section, students have the opportunity to apply their thinking from throughout the unit. As this is a short section followed by an End-of-Unit Assessment, there are no section goals or checkpoint questions.

Lessons

1. **Now What Can You Build? (Optional)** — Let’s construct some creative shapes.

Unit 2: Congruence

Unit Goal

Students prove that figures are congruent by reasoning about rigid transformations. They prove the Triangle Congruence Theorems (SSS, SAS, ASA), the Perpendicular Bisector Theorem, and several theorems about quadrilaterals.

Unit Narrative

In this unit, students prove a variety of figures congruent. They start with segments (2 vertices), then triangles (3 vertices), and finally quadrilaterals (4 vertices). Before starting this unit, students are familiar with rigid transformations and congruence from a previous unit. The triangle congruence theorems in this unit lay the foundation for triangle similarity in a subsequent unit.

The first section focuses on establishing congruence. Students recall from middle school that if two figures are congruent, then each pair of corresponding parts of the figures are also congruent. Next, students use rigid transformations to justify the triangle congruence theorems of Euclidean geometry: Side-Side-Side Triangle Congruence, Side-Angle-Side Triangle Congruence, and Angle-Side-Angle Triangle Congruence. Students justify that for each set of criteria, a sequence of rigid motions exists that will take one triangle onto the other.

In a previous unit, students began justifying their responses. In this unit, students write more rigorous proofs. At first, students may use imprecise language to convey their ideas. Throughout the unit they will read examples, practice explaining ideas to a partner, and build a reference of precise statements to use in future proofs. Writing proofs doesn't only mean providing the reasons for someone else's claim, constructing a viable argument includes writing conjectures and detailing given information.

Students write congruence proofs using transformations so the resulting theorems can be used in future work without repeating the argument. Many of the applications students explore involve quadrilaterals. Students will use the theorems they prove about quadrilaterals later in coordinate geometry as they use algebraic methods to prove additional results about quadrilaterals.

Section A: Congruent Figures

Section Goals

- Comprehend and generate congruence statements that establish corresponding parts.
- Justify whether or not figures are congruent by reasoning about rigid transformations.

Section Narrative

In this section students prove that if triangles are congruent, then all corresponding parts must be congruent, and they start developing the language they will use to prove triangle congruence theorems. As students progress through their study of proof, they have opportunities to create their own proofs as well as see models of increasingly formal language with which to express their reasoning. Creating a bank of statements and reasons for students to draw on helps reduce the cognitive demand of writing proofs with formal language so students can focus on the logical coherence of each new proof. In this section and in subsequent sections, students develop statements to use for proofs, such as "Since the vertices are the same distance along the same ray, they have to be in the same place. I know the points are the same distance from the endpoint because rigid transformations preserve distance." In addition to developing language for each part of a proof, students also develop theorems to help consolidate proofs. In this section students prove that any pair of segments with the same length must be congruent. This allows students to skip describing the sequence of transformations to line up one pair of sides in future proofs about congruent figures.

Lessons

1. **Congruent Parts, Part 1** — Let's figure out what the corresponding sides and angles in figures have to do with congruence.
2. **Congruent Parts, Part 2** — Let's name figures in ways that help us see the corresponding parts.
3. **Congruent Triangles, Part 1** — Let's use rigid transformations to be sure that two triangles are congruent.
4. **Congruent Triangles, Part 2** — Let's use rigid transformations to figure out if triangles are congruent.
5. **Points, Segments, and Zigzags** — Let's figure out when segments are congruent.

Section B: Triangle Congruence Theorems

Section Goals

- Identify situations where Side-Side-Side, Angle-Side-Angle, and Side-Angle-Side Triangle Congruence Theorems apply.
- Use previously proven theorems about triangles to write new proofs.

Section Narrative

In this section students prove the three Triangle Congruence Theorems and apply these theorems to prove other conjectures. Students begin this section with the skills to prove the Side-Angle-Side and Angle-Side-Angle triangle congruence theorems. They then pause to prove the Perpendicular Bisector Theorem in order to use it in the proof of the Side-Side-Side Triangle Congruence Theorem. There is an optional lesson for the ambiguous case of triangle congruence, Side-Side-Angle. Completing this optional lesson can help students better understand in which situations knowing two sides and an angle not between them defines a unique triangle. This lesson prepares students to understand when the law of sines and law of cosines might give ambiguous results when they study trigonometry in a future course. Applying the triangle congruence theorems often involves purposefully drawing auxiliary lines to make triangles with certain properties. Students practice this skill throughout the section.

Lessons

1. **Side-Angle-Side Triangle Congruence** — Let's use definitions and theorems to figure out what must be true about shapes, without having to measure all parts of the shapes.
2. **Angle-Side-Angle Triangle Congruence** — Let's see if we can prove other sets of measurements that guarantee triangles are congruent, and apply those theorems.
3. **The Perpendicular Bisector Theorem** — Let's convince ourselves that what we've conjectured about perpendicular bisectors must be true.
4. **Side-Side-Side Triangle Congruence** — Let's figure out if Side-Side-Side is enough information to guarantee triangles are congruent.
5. **Practicing Proofs** — Let's practice what we've learned about proofs and congruence.
6. **Side-Side-Angle (Sometimes) Congruence (Optional)** — Let's explore triangle congruence criteria that are ambiguous.

Section C: Proofs about Quadrilaterals

Section Goals

- Critique others' justifications about quadrilaterals and constructions.
- Use previously proven theorems to write new proofs about quadrilaterals.

Section Narrative

In this section students apply the triangle congruence theorems to write more proofs. They consider the diagonals of quadrilaterals, angle bisectors, and isosceles triangles. Students have the opportunity to engage in all aspects of the proof process: generating conjectures, detailing specific statements to be proved, writing a proof, and critiquing a proof. They continue creating auxiliary lines and encounter the challenge of overlapping triangles.

Lessons

1. **Proofs about Quadrilaterals** — Let's prove theorems about quadrilaterals and their diagonals.
2. **Proofs about Parallelograms** — Let's prove theorems about parallelograms.
3. **Bisect It** — Let's prove that some constructions we conjectured about really work.

Section D: Let's Put It to Work

Section Narrative

In this final section, students have the opportunity to apply their thinking from throughout the unit. As this is a short section followed by an End-of-Unit Assessment, there are no section goals or checkpoint questions.

Lessons

1. **Congruence for Quadrilaterals** — Let's investigate how congruence for quadrilaterals is similar to and different from congruence for triangles.

Unit 3: Similarity

Unit Goal

Students extend rigid transformations to include dilations and define similar figures. They prove the Angle-Angle Triangle Similarity Theorem, use similar right triangles to prove the Pythagorean Theorem and its converse, and use similarity to solve problems.

Unit Narrative

In this unit, students explore similar figures as pairs of figures for which one can be taken onto the other through a sequence of rigid transformations and a dilation. The work of this unit expands upon students' understanding of similarity, previously encountered in grade 8, and their work in an earlier unit on congruence.

The unit begins with an exploration of dilation properties, including the idea that angles are preserved through dilation and lines are taken to themselves or parallel lines depending on the center of dilation. Then students practice using a dilation along with rigid transformations to show that a pair of figures are similar and look for other ways to know that a pair of figures will be similar. The unit then focuses more closely on similar triangles, introducing theorems such as the Angle-Angle Similarity Theorem and connections to the Pythagorean Theorem.

Students focus on writing conjectures and proving them throughout the unit. Students should become more familiar with the process of noticing a pattern, making a conjecture, then looking to find counterexamples or justifying the conjecture with a proof.

Section A: Properties of Dilations

Section Goals

- Comprehend that dilations take angles to congruent angles and lines not passing through the center of the dilation to parallel lines.
- Create and interpret scaled drawings of figures.

Section Narrative

In this section, students review the concept of a dilation and expand upon their understanding to notice properties such as angles remaining congruent after dilation and lines being taken to the same or parallel lines after dilation. Students practice writing equivalent ratios of segment lengths before and after dilation to find unknown lengths after the same dilation. Then, they examine a particular case of similar triangles in which the center of the dilation is a vertex of the triangle.

Lessons

1. **Scale Drawings** — Let's make a scale drawing.
2. **Scale of the Solar System (Optional)** — Let's dilate figures.
3. **Measuring Dilations** — Let's dilate polygons.
4. **Dilating Lines and Angles** — Let's dilate lines and angles.
5. **Splitting Triangle Sides with Dilation (Part 1)** — Let's draw segments connecting midpoints of the sides of triangles.

Section B: Similarity Transformations and Proportional Reasoning

Section Goals

- Generate similarity statements about similar figures, and identify equivalent proportional relationships among side lengths.
- Use the Angle-Angle Triangle Similarity Theorem and the Triangle Angle Sum Theorem to show that two triangles are similar.

Section Narrative

In this section, students use the definition of similar figures to develop other conditions for guaranteeing similarity, especially in triangles. Students continue to develop their skills in making and proving conjectures by generalizing from a few examples. Students develop the Angle-Angle Triangle Similarity Theorem, and an optional lesson allows students to explore other possible conditions for triangle similarity theorems.

Lessons

1. **Connecting Similarity and Transformations** — Let's identify similar figures.
2. **Reasoning about Similarity with Transformations** — Let's describe similar triangles.
3. **Are They All Similar?** — Let's prove that figures are similar.
4. **Conditions for Triangle Similarity** — Let's prove that some triangles are similar.
5. **Other Conditions for Triangle Similarity (Optional)** — Let's prove that more triangles are similar.
6. **Splitting Triangle Sides with Dilation (Part 2)** — Let's investigate parallel segments in triangles.
7. **Practice with Proportional Relationships (Optional)** — Let's find unknown values in proportional relationships.

Section C: Similarity in Right Triangles

Section Goals

- Use similarity, proportionality, and the Pythagorean Theorem to solve problems involving right triangles.

Section Narrative

In this section, students explore similar right triangles which allow for the use of the Pythagorean Theorem in addition to the properties of similar triangles. They also use the properties of similar triangles to prove the Pythagorean Theorem, then practice using proportional relationships and the Pythagorean Theorem together to find unknown lengths.

Lessons

1. **Using the Pythagorean Theorem and Similarity** — Let's explore right triangles with altitudes drawn to the hypotenuse.
2. **Proving the Pythagorean Theorem** — Let's prove the Pythagorean Theorem.
3. **Converse of the Pythagorean Theorem** — Let's look at the converse of the Pythagorean Theorem.
4. **Finding All the Unknown Values in Triangles** — Let's find all the unknown values in right triangles.

Section D: Let's Put It to Work

Section Narrative

In this final section, students have the opportunity to apply their thinking from throughout the unit. As this is a short section followed by an End-of-Unit Assessment, there are no section goals or checkpoint questions.

Lessons

1. **Reflection Similarity** — Let’s use similarity to solve problems.

Unit 4: Right Triangle Trigonometry

Unit Goal

Students build an understanding of ratios in right triangles, naming cosine, sine, and tangent as trigonometric ratios. They use trigonometry to find missing side lengths and angle measures and apply trigonometry to solve problems in context.

Unit Narrative

In this unit students build an understanding of ratios in right triangles, which leads to naming cosine, sine, and tangent as trigonometric ratios.

Prior to beginning this unit, students will have considerable familiarity with right triangles and similarity. They learned to identify right triangles in grade 4. Students studied the Pythagorean Theorem in grade 8, and used similar right triangles to build the idea of slope. This unit builds on this extensive experience and grounds trigonometric ratios in familiar contexts.

Several concepts build throughout the unit. Students begin by using similar triangles to create a table of ratios of the side lengths in right triangles. Taking the time to both build and use the table helps students construct a solid foundation before they learn the names of trigonometric ratios.

Students notice patterns between the columns for cosine and sine before they first hear the terms "cosine" and "sine." In a subsequent lesson they investigate that relationship, proving the two ratios are equal for complementary angles. Finding the measures of acute angles in a right triangle follows a similar arc, where students first use the table to estimate and then in a subsequent lesson learn how to calculate an angle measure given the side measures by using arcsine, arccosine, and arctangent.

As students measure side lengths and compute ratios, there is an opportunity to discuss precision. In this unit, students will round side lengths to the nearest tenth and angle measures to the nearest degree in most cases. When students solve problems in context they grapple with whether or not their answer is reasonable, as well as the appropriate degree of precision to report.

Section A: Angles and Steepness

Section Goals

- Comprehend that one acute angle of a right triangle determines all the ratios of the side lengths in that triangle.
- Justify an estimated side length or angle measure using a table of ratios of side lengths of right triangles.

Section Narrative

In this section, students work with right triangles to build the foundation for trigonometry. They practice describing similar right triangles to solidify the idea that any right triangles with congruent acute angles are similar. Students extend this understanding when they generate data for the ratios of the side lengths of many sets of right triangles. These data are organized into a table that students apply to problems. Taking the time to both build and use the table helps students establish a solid foundation before they learn the names of trigonometric ratios. Two of the lessons in this section are optional since the standards do not specifically call for special right triangles. These lessons are an opportunity to practice the important ideas with special right triangles, which are frequently included on college entrance exams.

Lessons

1. **Angles and Steepness** — Let's solve problems about right triangles.
2. **Half a Square (Optional)** — Let's investigate the properties of diagonals of squares.
3. **Half an Equilateral Triangle (Optional)** — Let's investigate the properties of altitudes of equilateral triangles.
4. **Ratios in Right Triangles** — Let's investigate ratios in the side lengths of right triangles.
5. **Working with Ratios in Right Triangles** — Let's solve problems about right triangles.

Section B: Defining Trigonometric Ratios

Section Goals

- Calculate angle measures in right triangles using arccosine, arcsine, and arctangent.
- Calculate side lengths in right triangles using cosine, sine, and tangent.
- Use trigonometry to solve problems in context.

Section Narrative

In this section, students learn to use trigonometry to solve problems. They learn that the names "cosine," "sine," and "tangent" correspond to the columns of their table of ratios and practice looking up the cosine, sine, or tangent of a given angle in a calculator. They use these values to find the side lengths of right triangles. Next, students investigate the relationship between cosine and sine, proving the two ratios are equal for complementary angles. Finally, students learn how to find an angle measure of a right triangle by using the given side lengths and arcsine, arccosine, and arctangent. Throughout the section students apply their trigonometry skills to several contexts.

Lessons

1. **Working with Trigonometric Ratios** — Let's solve problems using cosine, sine, and tangent.
2. **Applying Ratios in Right Triangles** — Let's solve problems by using right triangles and trigonometry.
3. **Sine and Cosine in the Same Right Triangle** — Let's connect cosine and sine.
4. **Trigonometry Squared** — Let's connect the squares of some trigonometric ratios.
5. **Using Trigonometric Ratios to Find Angles** — Let's work backwards to find angles in right triangles.

Section C: Let's Put It to Work

Section Narrative

In this final section, students have the opportunity to apply their thinking from throughout the unit. As this is a short section followed by an End-of-Unit Assessment, there are no section goals or checkpoint questions.

Lessons

1. **Solving Problems with Trigonometry** — Let's solve problems about right triangles.
2. **Approximating Pi** — Let's approximate the value of pi.

Unit 5: Solid Geometry

Unit Goal

Students practice spatial visualization in three dimensions. They sketch cross-sections, study how surface area and volume scale under dilation, derive volume formulas for prisms, cylinders, pyramids, and cones using Cavalieri's Principle, and apply their understandings to solve problems.

Unit Narrative

In this unit, students practice spatial visualization in three dimensions. They sketch cross-sections, study dilations, derive volume formulas, and apply their understandings to solve problems.

In previous grades, students studied multiple aspects of solids. In grade 6, they started calculating surface areas and volumes of right rectangular prisms. They extended this understanding to the volume and surface area of right prisms in grade 7, and to that of spheres, cones, and cylinders in grade 8. Students also described cross-sections of three-dimensional figures in grade 7.

In future courses, the visualization skills developed in this unit will be applicable when using calculus to compute volumes or when using linear algebra in three or more dimensions.

Students begin the unit by examining solids of rotation and cross-sections of a variety of solids. They make a connection between cross-sections and dilation to see that cross-sections of a pyramid may be viewed as dilations of the base for scale factors between 0 and 1. Later in the unit they learn Cavalieri's Principle: Suppose two solids have equal heights. If at all distances from the base, the cross-sections of the two solids have equal areas, then the solids have equal volumes. Combining the concepts of dilations, cross-sections, and Cavalieri's Principle with dissection allows students to derive the volume formulas for a pyramid or cone.

When students establish that dilating by a scale factor of k multiplies areas by k^2 and volumes by k^3, it provides an opportunity to use square root and cube root graphs to illustrate the relationship between scaled area or volume and scale factors. While this unit is the primary opportunity to study root functions, students will continue to graph functions and interpret the meaning of points, coefficients, and constants in future courses.

In this unit, students may assume that cylinders or prisms that appear to be oblique are indeed oblique, and those that appear to be right are right. For right cylinders and prisms, right angles will not be marked to indicate that the bases are at right angles to the lateral surfaces. Unless otherwise stated, all responses given in decimal form are rounded to the nearest tenth.

Section A: Cross-Sections, Scaling, and Area

Section Goals

- Describe the relationships between scale factors and areas, using square root graphs and calculations.
- Generate multiple cross-sections of three-dimensional figures.
- Identify the three-dimensional solid created by rotating a two-dimensional figure, using a linear axis.

Section Narrative

In this section, students practice spatial visualization. First they do this by examining solids of rotation. Then they investigate cross-sections of a variety of solids. They create physical representations to show that cross-sections of a pyramid may be viewed as dilations of the base for scale factors that are between 0 and 1. Students study the effect of dilation on cross-sections and other two-dimensional figures, establishing that dilating by a scale factor of k multiplies areas by k^2. They're introduced to the equation $y = x^2$ in this geometric context. They create a graph representing $y = \sqrt{x}$ and use it to illustrate the relationship between scaled area and scale factors.

Lessons

1. **Solids of Rotation** — Let's rotate two-dimensional shapes to make three-dimensional shapes.
2. **Slicing Solids** — Let's analyze cross-sections by slicing three-dimensional solids.
3. **Creating Cross-Sections by Dilating** — Let's create cross-sections by doing dilations.
4. **Scaling and Area** — Let's see how the area of shapes changes when we dilate them.
5. **Scaling and Unscaling** — Let's examine the relationships between areas of dilated figures and scale factors.

Section B: Scaling Solids

Section Goals

- Comprehend that when a solid is dilated by a scale factor of k, its surface area is multiplied by k^2 and its volume is multiplied by k^3.
- Describe the relationships between scale factors and volumes using cube root graphs and calculations.

Section Narrative

In this section, students extend their study of scaling to solids. They conclude that dilating a solid by a scale factor of k multiplies all lengths by k, surface areas by k^2, and volumes by k^3. They work backward from a scaled surface area or volume to find the scale factor involved, which requires the introduction of cube roots. Students create a graph representing $y = \sqrt[3]{x}$ and use it to answer questions about how changes in volume affect changes in the corresponding scale factor.

Lessons

1. **Scaling Solids** — Let's see how the surface area and volume of solids change when we dilate those solids.
2. **The Root of the Problem** — Let's look at relationships between volumes, areas, and scale factors using graphs and situations.
3. **Speaking of Scaling** — Let's practice moving back and forth between scale factors for lengths, surface areas, and volumes.

Section C: Prism and Cylinder Volumes

Section Goals

- Calculate volumes of solids composed of right and oblique prisms and cylinders.
- Explain connections between ways of finding volumes of prisms and cylinders.

Section Narrative

In this section students revisit the volume of a cylinder from previous grade levels, and solids of rotation from previous lessons. Then students are introduced to Cavalieri's Principle: Suppose two solids have equal heights. If, at all distances from the base, the cross-sections of the two solids have equal areas, then the solids have equal volumes. This leads to the idea that the volume of a prism or cylinder with a base area of B square units and a height of h units has a volume of Bh cubic units, regardless of the shape of the base and regardless of whether the solid is oblique. If necessary here and in the remainder of the unit, provide students with formulas for the areas of triangles, parallelograms, and circles. Note that “area of a circle” is used as shorthand for “the area of the region enclosed by a circle.”

Lessons

1. **Cylinder Volumes** — Let’s analyze cylinder volumes.
2. **Cross-Sections and Volume** — Let’s look at how cross-sections and volume are related.
3. **Prisms Practice** — Let’s calculate volumes of prisms and cylinders.

Section D: Understanding Pyramid Volumes

Section Goals

- Calculate volumes and dimensions of prisms, cylinders, cones, and pyramids.
- Use decomposition and Cavalieri’s Principle to informally justify the volume formula for pyramids.

Section Narrative

In this section, students combine concepts of dilations, cross-sections, and Cavalieri’s Principle with dissection to derive the formula for the volume of a pyramid or cone. First, they establish that any triangular pyramid whose base has area B square units and whose height is h units can be combined with two other triangular pyramids of equal volume to form a prism with the same base and height as the original pyramid. Each pyramid has ⅓ the volume of the prism. Therefore, the volume of the original pyramid is ⅓Bh. Next, students consider a general pyramid, and compare it to a triangular pyramid with equal height and a base of equal area. Because all corresponding cross-sections have equal area, Cavalieri’s Principle applies, and the two pyramids have equal volume. This extends the volume formula to all pyramids and cones, regardless of the particular shape of the base or whether the solid is oblique. Students then have the opportunity to practice applying volume formulas to a variety of problems. These activities include opportunities for algebraic manipulation and aspects of mathematical modeling.

Lessons

1. **Prisms and Pyramids** — Let’s describe relationships between pyramids and prisms.
2. **Building a Volume Formula for a Pyramid** — Let’s create a formula for the volume of any pyramid or cone.
3. **Working with Pyramids** — Let’s use the pyramid volume formula to solve problems.
4. **Putting All the Solids Together** — Let’s calculate volumes of prisms, cylinders, cones, and pyramids.

Section E: Let’s Put It to Work

Section Narrative

In the final section, students have the opportunity to apply their thinking from throughout the unit. Because this is a short section followed by an End-of-Unit Assessment, there are no section goals or checkpoint questions.

Lessons

1. **Surface Area and Volume** — Let’s use volume and surface area to solve problems.
2. **Volume and Density** — Let’s use volume and density to solve problems.

3. **Volume and Graphing** — Let’s use volume and graphing to solve problems.

Unit 6: Coordinate Geometry

Unit Goal

Students bring algebra and geometry together by working with transformations as functions, equations of circles and parabolas, equations of lines (parallel and perpendicular), coordinate proofs of geometric theorems, and weighted averages.

Unit Narrative

Prior to beginning this unit, students will have spent most of the course studying geometric figures not described by coordinates. However, students have seen figures on the grid (notably transformations in grade 8) as well as lines and curves on the coordinate plane (in previous courses). This unit brings together students' experience from previous years with their new understanding from this course for an in-depth study of coordinate geometry.

The first few lessons examine transformations in the plane. Students encounter a new coordinate transformation notation that connects transformations to functions. Students transform figures using rules such as $(x, y) \rightarrow (x+a, y+b)$ and connect the geometric definitions of reflections and dilations to coordinate rules that produce them. They prove that objects are similar or congruent, using reasoning including distance (via the Pythagorean Theorem), angles (calculated using trigonometry), and definitions of transformations.

The next set of lessons focuses on building equations from definitions. Students examine circles and parabolas through the lens of distance. A circle is the set of points the same distance from a given center, and a parabola is the set of points equidistant from a given point (the focus) and line (the directrix). Based on these definitions, students develop a general equation for a circle, and they write equations that represent specific parabolas.

The unit progresses next to coordinate proof. Students build the point-slope form of the equation of a line. They then write and prove conjectures about slopes of parallel and perpendicular lines, applying concepts of transformations in the proofs. They apply these ideas to other proofs, such as classifying quadrilaterals, and they use graphs to solve simple systems of equations that include a linear equation and a quadratic equation.

At the end of the unit, students use weighted averages to partition segments, scale figures, and locate the intersection points of the medians of a triangle. Students locate the intersection points of the altitudes of a triangle. Then there are several optional activities that offer diverging paths toward the Euler line or toward practicing equations of lines through constructing and describing tessellations.

In the final lesson, students apply their understanding of slope and distances in a plane, as they explore the Nazca lines in a real-world situation.

Section A: Transformations in the Plane

Section Goals

- Compare and contrast rigid transformations, similarity transformations, and those that are neither.
- Describe transformations as functions that take points in the plane as inputs and give other points as outputs.

Section Narrative

This section focuses on understanding transformations as a type of function. Students first recall work from earlier units as they connect rigid transformations in the coordinate plane using the Pythagorean Theorem to identify lengths of segments. Students are then introduced to function notation for transformations as they identify $(x, y) \rightarrow (x+a, y+b)$ as a translation, $(x, y) \rightarrow (kx, ky)$ as a dilation with scale factor k, and transformations like $(x, y) \rightarrow (x, -y)$ as a reflection. They also use tables and figures to create their own function rules to represent transformations in the coordinate plane. Finally, students compare transformation functions and decide whether they represent rigid transformations, similarity transformations, or neither.

Lessons

1. **Rigid Transformations in a Plane** — Let's try transformations with coordinates.
2. **Transformations as Functions** — Let's compare transformations to functions.
3. **Types of Transformations** — Let's analyze transformations that produce congruent and similar figures.

Section B: Distances, Circles, and Parabolas

Section Goals

- Calculate the center and radius of a circle by completing the square.
- Use the Pythagorean Theorem to write an equation for a parabola given its focus and directrix.

Section Narrative

This section focuses on the features of equations and graphs of circles and parabolas. In previous units, students defined a circle as the set of points equidistant from a given center. They connect this definition to their work with the Pythagorean Theorem in order to generalize an equation of a circle with a radius r and a center (h, k) as $(x-h)^2 + (y-k)^2 = r^2$. Next, students revisit the distributive property as they write a squared binomial as a perfect square trinomial and vice versa. Then, students determine constants needed to complete the square for trinomial expressions and rewrite an equation for a circle in the form $x^2 + y^2 + ax + by + c = 0$ in order to determine its center and radius. Next, students examine the graphical properties of a parabola, including the focus and directrix. They observe that any point on the parabola is the same distance from the focus and the directrix, and that this property can be used to define a parabola. Finally, students use the Pythagorean Theorem to calculate the distance between a point and the focus, and compare it to the vertical distance between that point and the directrix. Students use the fact that these distances must be equal when they write a general equation for a parabola where the directrix is a horizontal line.

Lessons

1. **Distances and Circles** — Let's build an equation for a circle.
2. **Squares and Circles** — Let's see how the distributive property can relate to equations of circles.
3. **Completing the Square** — Let's rewrite equations to find the center and radius of a circle.
4. **Distances and Parabolas** — Let's analyze the set of points that are the same distance from a given point and a given line.
5. **Equations and Graphs** — Let's write an equation for a parabola.

Section C: Proving Geometric Theorems Algebraically

Section Goals

- Calculate the coordinates of a point on a line segment that partitions the segment in a given ratio.
- Calculate the intersection point of a triangle's medians.
- Determine equations for lines, through a given point, that are parallel or perpendicular to a given line.

Section Narrative

In this section, students focus on proving geometric theorems algebraically. First, students connect the point-slope form of a line to their work with translations of lines. They then build on this work to prove that parallel lines have the same slope. Students then prove that a quadrilateral with opposite sides that lie on two pairs of parallel lines must be a parallelogram. Next, students prove that the slopes of perpendicular lines are opposite reciprocals, or that the product of the slopes must be -1. Then students apply theorems about parallel and perpendicular lines to show that two noncollinear lines that are perpendicular to the same line are parallel to each other. Students then inspect curves in the coordinate plane, as they examine the intersection points of circles and lines. They use the Pythagorean Theorem to find points that lie on both the line and the circle. Then they apply theorems about slope and the Pythagorean Theorem to categorize quadrilaterals in the plane. Next, students examine ratios, as they partition segments using weighted averages. They also apply this method to determine a coordinate definition of dilation. Finally, students have an opportunity to apply work with slopes and the Pythagorean Theorem as they explore additional triangle theorems.

Lessons

1. **Equations of Lines** — Let's investigate equations of lines.
2. **Parallel Lines in the Plane** — Let's investigate parallel lines in the coordinate plane.
3. **Perpendicular Lines in the Plane** — Let's analyze the slopes of perpendicular lines.
4. **It's All on the Line** — Let's work with both parallel and perpendicular lines.
5. **Intersection Points** — Let's look at how circles and parabolas interact with lines.
6. **Coordinate Proof** — Let's use coordinates to prove theorems and to compute perimeter and area.
7. **Weighted Averages** — Let's split segments using averages and ratios.
8. **Weighted Averages in a Triangle** — Let's partition special line segments in triangles.
9. **Lines in Triangles** — Let's investigate more special segments in triangles.

Section D: Let's Put It to Work

Section Narrative

In this final section, students have the opportunity to apply their thinking from throughout the unit. As this is a short section followed by an End-of-Unit Assessment, there are no section goals or checkpoint questions.

Lessons

1. **Applying Area and Perimeter on the Plane** — Let's use coordinate geometry with a real-world situation.

Unit 7: Circles

Unit Goal

Students investigate the geometry of circles, including chords, arcs, central and inscribed angles, tangents, cyclic quadrilaterals, inscribed and circumscribed circles of triangles, sector area, arc length, and radian measure.

Unit Narrative

In this unit, students investigate the geometry of circles more closely. In grade 7, students used formulas for the area and circumference of a circle to solve problems. Earlier in this course, students made formal geometric constructions, studied similarity and proportional reasoning, and proved theorems about lines and angles. This unit builds on these skills and concepts. In Algebra 2, the concepts learned in this unit will be helpful as students connect the unit circle to trigonometric functions.

Students define the terms “chord,” “arc,” and “central angle” before observing that inscribed angles are half the measure of their associated central angles, and writing related proofs about congruent chords and similar triangles. Throughout this unit, students also construct lines tangent to circles and use their proofs that a tangent line is perpendicular to the radius drawn to the point of tangency.

Next, students prove properties of cyclic quadrilaterals, and they use their understanding of perpendicular bisectors from a previous unit to construct triangles with circumscribed circles and define “circumcenter.” Students then use angle bisectors to construct incenters of triangles and circles inscribed in triangles.

In the next section, students develop methods for calculating sector areas and arc lengths, and then students define “radian measure of a central angle” as the quotient of the length of the arc defined by the angle and the radius of the circle. They develop fluency with radian measures by shading portions of circles and working with a double number line.

In the final lesson, students apply what they have learned about circles to solve problems in context.

In this unit, students will do several constructions. A particular choice of construction tools is not necessary. Paper folding and straightedge and compass moves are both acceptable methods.

Section A: Lines, Angles, and Circles

Section Goals

- Determine measures of central angles, inscribed angles, and arcs.
- Use the relationship between radii, chords, and tangent lines to prove geometric theorems.

Section Narrative

In this section, students encounter new types of angles and mathematical objects defined by circles. First, students write definitions of “chord,” “central angle,” and “arc,” and use their new understanding to write a proof about congruent chords in circles. Students then use those definitions to make conjectures about properties of inscribed angles and develop an understanding of the Inscribed Angle Theorem. Finally, students prove statements about lines tangent to a circle and analyze relationships between central angles, tangent lines, and angles circumscribed about a circle.

Lessons

1. **Lines, Angles, and Curves** — Let’s define some line segments and angles related to circles.
2. **Inscribed Angles** — Let’s analyze angles made from chords.

3. **Tangent Lines** — Let’s explore lines that intersect a circle at exactly 1 point.

Section B: Polygons and Circles

Section Goals

- Construct the inscribed and circumscribed circles of a triangle.
- Prove properties of the angles in a quadrilateral inscribed in a circle.

Section Narrative

In this section, students learn to use circles to determine characteristics of triangles and other polygons. First, students work with circumscribed circles to find angle measures and properties of cyclic quadrilaterals. Students then use the circumscribed circle of a triangle, along with their understanding of perpendicular bisectors from previous units, to conclude that all three of the triangle’s perpendicular bisectors intersect at its circumcenter. Finally, students analyze characteristics of angle bisectors, define “incenter” (of a triangle), and construct inscribed circles.

Lessons

1. **Quadrilaterals in Circles** — Let’s investigate quadrilaterals that fit in a circle.
2. **Triangles in Circles** — Let’s see how perpendicular bisectors relate to circumscribed circles.
3. **A Special Point** — Let’s see what we can learn about a triangle by watching how salt piles up on it.
4. **Circles in Triangles** — Let’s construct the largest possible circle inside of a triangle.

Section C: Measuring Circles

Section Goals

- Comprehend that the radian measure of an angle whose vertex is the center of a circle is the ratio of the length of the arc defined by the angle to the circle’s radius.
- Coordinate between representations of angles measured in degrees and in radians.
- Generalize methods for calculating lengths of arcs and areas of sectors in circles.

Section Narrative

In this section, students learn the relationships between measures of some different parts of circles. First, students analyze circle sectors and find sector areas and arc lengths. Students then investigate the relationships between angle measure, radius, and arc length to find unknown information about given circles. Next, students experience a progression of learning about the relationship between arc length and central angle that builds to a definition of “radians.” Using their understanding of radians, students finally build their intuition about the size of a radian by comparing measures in both degrees and radians.

Lessons

1. **Arcs and Sectors** — Let’s analyze portions of circles.
2. **Part to Whole** — Let’s see what we can figure out about a circle if we’re given information about a sector of the circle.
3. **Angles, Arcs, and Radii** — Let’s analyze relationships between arc lengths, radii, and central angles.
4. **A New Way to Measure Angles** — Let’s look at a new way to measure angles.

5. **Radian Sense** — Let’s get a sense for the sizes of angles measured in radians.
6. **Using Radians** — Let’s see how radians can help us calculate sector areas and arc lengths.

Section D: Let’s Put It to Work

Section Narrative

In this final section, students have the opportunity to apply their thinking from throughout the unit. As this is a short section followed by an End-of-Unit Assessment, there are no section goals or checkpoint questions.

Lessons

1. **Putting It All Together** — Let’s use geometry to solve problems.

Unit 8: Conditional Probability

Unit Goal

Students extend their understanding of probability to compound events. They use lists, tables, trees, and Venn diagrams to represent sample spaces, apply the Addition Rule and the Multiplication Rule, define conditional probability, and use probability to test events for independence.

Unit Narrative

In this unit, students extend their understanding of probability, sample spaces, and events from their introduction in grade 7. The chance experiments under consideration have multiple parts, such as rolling a number cube and then flipping a coin—allowing events within the sample space to be considered in new ways.

The unit begins with students creating different models for understanding sample spaces and probability. The models include tables, trees, lists, and Venn diagrams. Venn diagrams allow students to visualize various subsets of the sample space, such as "A and B," "A or B," or "not A." Tables help students determine the probability of those subsets occurring, and support students' understanding of the Addition Rule, P(A or B) = P(A) + P(B) - P(A and B).

Conditional probability is discussed and applied using several games and connections to everyday situations. In particular, the Multiplication Rule P(A and B) = P(A) · P(B | A) is used to determine conditional probabilities. Conditional probability leads to a study of independence of events. Students describe independence using everyday language and use the equations P(A | B) = P(A) and P(A and B) = P(A) · P(B) when events A and B are independent.

The unit closes with students making conjectures about the independence of events, when playing games with one another, and then testing those conjectures by collecting data and analyzing the results.

Section A: Up to Chance

Section Goals

- Use organized lists, tables, and tree diagrams to calculate probabilities for compound events.
- Use the addition rule to calculate probabilities.

Section Narrative

In this section, students revisit probability of events. They are reminded of the connection between the probability of an event and the proportion of times a chance experiment results in that event after multiple trials. Then students look at compound events and explore ways to keep track of events in the sample space using lists, tables, trees, and Venn diagrams. Students can then draw connections between different combinations of events through the Addition Rule P(A or B) = P(A) + P(B) - P(A and B). An optional lesson is included at the beginning of the section if students could benefit from additional reminders about probability.

Lessons

1. **Up to Chance (Optional)** — Let's explore chance.
2. **Playing with Probability** — Let's explore probability.
3. **Sample Spaces** — Let's look closer at sample spaces.
4. **Tables of Relative Frequencies** — Let's use tables to organize probabilities.
5. **Combining Events** — Let's look at ways to describe events composed of other events.

6. **The Addition Rule** — Let’s learn about and use the addition rule.

Section B: Related Events

Section Goals

- Determine whether two events are independent or dependent using probability.
- Use the multiplication rule, P(A and B) = P(A) · P(B | A), to find the probability of an event.

Section Narrative

In this section, students look at pairs of events to determine whether they are dependent or independent events. After defining the terms, students do some classic statistics experiments and practice using their understanding of dependent events to explain why the results make sense. Then students encounter the multiplication rule P(A and B) = P(A) · P(B | A) and later use it to test for independence of events.

Lessons

1. **Related Events** — Let’s see how events are related.
2. **Conditional Probability** — Let’s examine conditional probability.
3. **Using Tables for Conditional Probability** — Let’s use tables to estimate conditional probabilities.
4. **Using Probability to Determine Whether Events Are Independent** — Let’s take a closer look at dependent and independent events.

Section C: Let’s Put It to Work

Section Narrative

In this final section, students have the opportunity to apply their thinking from throughout the unit. Because this is a short section followed by an End-of-Unit Assessment, there are no section goals or checkpoint questions.

Lessons

1. **Probabilities in Games (Optional)** — Let’s use probability with games.

About This Syllabus

This syllabus was compiled from the publicly available Illustrative Mathematics 9-12 AGA v.360 curriculum at accessim.org. Unit and section narratives, goals, and lesson titles are sourced from the teacher view. Optional lessons are labeled “(Optional).”
Assessments (pre-unit checks, checkpoints, mid-unit and end-of-unit assessments) are referenced in the source curriculum but require a Kendall Hunt educator account to view; they are not included here.
https://accessim.org/9-12-aga/geometry?a=teacher