Algebra 2 · Unit 5 · Assessment Guide

Exponential Functions and Equations

For the Parent or Teacher — What to look for

Unit 5 develops exponential functions and introduces logarithms as the INVERSE operation for finding an unknown exponent. The deepest diagnostic is whether students see log_b(x) = y and b^y = x as the SAME relationship in different clothes.

Problem 1 — Bacteria growth

- (a) P(t) = 200 · 3^t.
- (b) P(5) = 200 · 243 = 48,600. P(8) = 200 · 6561 = 1,312,200.
- (c) P(0.5) = 200 · √3 ≈ 346.4. ★ The diagnostic: do they accept that the function STILL works for non-integer t? Earlier they may have thought 3^0.5 was undefined.
- Strong: "3^0.5 = √3 because that's what the rule says: 3^(1/2) is the square root of 3."

Problem 2 — Annual ⇔ monthly rate

- (a) V(t) = 1000 · 1.08^t.
- (b) m = 1.08^(1/12) ≈ 1.00643. So monthly growth ≈ 0.643%.
- (c) ★ The diagnostic: 0.643% is NOT the same as 8/12 ≈ 0.667%. Compound growth is sub-linear in the rate — you don't just divide.
- Strong: "Dividing assumes growth adds; exponential growth MULTIPLIES, so we need the 12th root."

Problem 3 — Logs without a calculator

- $\log_2(32) = 5$. $\log_2(1) = 0$. $\log_2(1/8) = -3$.
- $\log_{10}(1000) = 3$. $\log_{10}(0.01) = -2$.
- $\log_3(81) = 4$. $\log_5(\sqrt{5}) = 1/2$.
- (d) ★ THE definition: log_b(x) is the exponent you put on b to get x. Strong students phrase it this way.
- Most diagnostic: $\log_5(\sqrt{5}) = 1/2$ — students who don't see √5 = 5^(1/2) get stuck.

Problem 4 — Solving exponentials

- (a) x = 6 (since 2^6 = 64).
- (b) $x = \log_3(20) = \log(20)/\log(3) \approx 2.727$.
- (c) 2^t = 20, $t = \log_2(20) \approx 4.322$.
- (d) $2x + 1 = \log_{10}(500) \approx 2.699$; x ≈ 0.850.
- ★ The pattern: take a log of both sides, then solve linearly. Students who can't 'see' when to introduce a log are stuck at the heart of the unit.

Problem 5 — Approaching e

- (n=100) ≈ 2.7048. (n=1000) ≈ 2.7169. (n=1,000,000) ≈ 2.71828.
- Approaches e ≈ 2.71828...
- (c) Strong: "e is the limit of continuously compounded growth. When growth is happening every instant (not just every period), the multiplier is e, not (1 + r)^n."
- ★ This problem is the bridge between discrete compounding and continuous (natural) growth.

Problem 6 — Half-life modeling

- (a) $A(t) = 100 \cdot (1/2)^{(t/12)}$.
- (b) $25 = 100 \cdot (1/2)^{(t/12)} \Rightarrow (1/2)^{(t/12)} = 1/4 = (1/2)^2$. $\Rightarrow t/12 = 2 \Rightarrow t = 24$ years.
- (c) $10 = 100 \cdot (1/2)^{(t/12)} \Rightarrow (1/2)^{(t/12)} = 0.1 \Rightarrow t/12 = \log_{0.5}(0.1) = \ln(0.1)/\ln(0.5) \approx 3.322 \Rightarrow t \approx 39.86$ years.
- ★ Watch for: students who set up $A(t) = 100 \cdot 0.5^t$ (forgetting to scale t by half-life).

Problem 7 — Two log bases

- Method 1: $\log(4^x) = \log(7) \Rightarrow x \cdot \log(4) = \log(7) \Rightarrow x = \log(7)/\log(4) \approx 1.404$.
- Method 2: $\ln(4^x) = \ln(7) \Rightarrow x \cdot \ln(4) = \ln(7) \Rightarrow x = \ln(7)/\ln(4) \approx 1.404$.
- ★ THE diagnostic: identical answer because the change-of-base formula. $\log_a(N)/\log_a(M)$ is independent of which base a you choose (within a single division).
- Strong: "Logs are just a way to count exponents; the relative size is fixed regardless of base."

Problem 8 — Diego's log rule error

- WRONG. Test: $\log(10 \cdot 100) = \log(1000) = 3$. $\log(10) \cdot \log(100) = 1 \cdot 2 = 2$. Not equal.
- Correct rule: $\log(a \cdot b) = \log(a) + \log(b)$ — logs turn multiplication into addition.
- ★ The most common log error in Algebra 2. Strong students see that the rule for products is ADDITION, not multiplication, because logs are exponents and exponents add when bases match.

Problem 9 — Exponential vs. polynomial growth

- TRUE. Any exponential function b^x with $b > 1$ eventually exceeds any polynomial $p(x)$, no matter the degree.
- Example: 2^x vs x^{10}. For small x, x^{10} wins. By $x \approx 60$, 2^x has caught up. By $x = 100$, 2^x is astronomically larger.
- ★ Strong students explain the underlying reason: exponential growth COMPOUNDS, polynomial growth is just power-of-input. Compounding always wins given enough x.
- Weak: states the fact without an example.

Problem 10 — Real-world logs

- ★ Open-ended; strong answers connect logs to compressing large dynamic ranges:
- pH: log of hydrogen ion concentration; a pH change of 1 means 10× difference.
- Richter scale: log of seismic amplitude; magnitude 6 is 10× stronger than magnitude 5.
- Decibels: log of sound power ratio.
- ★ The key insight: logs let humans manage quantities spanning many orders of magnitude on a manageable scale.

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